

# Convergence of meshfree collocation methods for fully nonlinear parabolic equations

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## Abstract

We prove the convergence of meshfree collocation methods for the terminal value problems of linear and fully nonlinear parabolic partial differential equations in the framework of viscosity solutions, provided that the basis function approximations of the terminal condition and the nonlinearities are successful at each time step. A numerical experiment with a radial basis function demonstrates the convergence property.

**Key words:** meshfree methods, parabolic equations, viscosity solutions, radial basis functions.

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## 1 Introduction

In this paper, we are concerned with the numerical methods for the terminal value problems of the parabolic partial differential equations:

$$(1.1) \quad \begin{cases} -\partial_t v + F(t, x, v(t, x), Dv(t, x), D^2v(t, x)) = 0, & (t, x) \in [0, T) \times \mathbb{R}^d, \\ v(T, x) = f(x), & x \in \mathbb{R}^d, \end{cases}$$

where  $F : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \rightarrow \mathbb{R}$ , and  $\mathbb{S}^d$  stands for the totality of symmetric  $d \times d$  real matrices. Here we have denoted by  $\partial_t$  the partial differential operator with respect to the time variable  $t$ , by  $D^j \equiv D_x^j$  the  $j$ -th order partial differential operator with respect to the space variable  $x$ . The conditions imposed on the function  $F$  is described in Section 2 below. The terminal value problem (1.1) mainly appears from probabilistic problems. In linear cases the solution to (1.1) is given by the expectation of a diffusion process, whereas in nonlinear cases of Hamilton-Jacobi-Bellman type, the solution is given by the value function of a stochastic control problem.

Existing numerical methods applicable to (1.1) are the finite difference methods (see, e.g., Kushner and Dupuis [11] and Bonnans and Zidani [2]), the finite-element like methods (see, e.g., Camilli and Falcone [3] and Debrabant and Jakobsen [5]), and the probabilistic methods (see, e.g., Pagès et al. [12] and Fahim et al. [7]). It should be mentioned that these methods have difficulties in applying to the problems with high-dimensional state space, which appear as an application of (1.1). For examples, in the finite difference methods, the diffusion matrix in the Hamiltonian should basically be diagonally dominant for ensuring its convergence (see, e.g., [11]). Also, the finite-element like methods require the interpolation of the solutions in the state space that preserve a monotonicity condition, and need involved computational procedures for the implementation in high-dimensional problems (see Carlini et al. [4]).

Another possible approach to (1.1) is to use the meshfree collocation method proposed by Kansa [9]. In this method, we seek an approximate solution of the form of a linear combination of a radial basis function (e.g., multiquadrics in the Kansa's original work). Substituting this form into a partial differential equation leads to an equation for the collocation points. Then the approximate solution is constructed by the meshfree interpolation of these collocation points. In general, this procedure allows for a simpler numerical implementation compared to the finite-element like methods, and it needs less computational time compared to the probabilistic methods. As for the convergence, Huang et al. [8] numerically shows the case of a Hamilton-Jacobi-Bellman equation of the first order, a special case of (1.1). However, to the best of our knowledge, the rigorous convergence issue for (1.1) has not been addressed in the literature.

In this paper, we aim to prove the rigorous convergence of the collocation method for (1.1). In doing so, we consider solutions of (1.1) in the viscosity sense since the smoothness of solutions cannot be expected in our nonlinear cases. In this framework, it is known that the abstract method proposed by Barles and Souganidis [1] is a powerful tool for checking the convergence of a given family of functions to a unique viscosity solution. Roughly speaking, if an operator that constructs the possible approximate solution has monotonicity, stability, and consistency properties, then by the arguments in [1] we can basically prove its convergence. In our case, however, this technique cannot be applied in a trivial way since the collocation method includes the derivative terms and thus violates the monotonicity condition. We find that a key to overcoming this difficulty is Lemma 4.1 in Kohn and Serfaty [10]. Using this lemma, they show that an approximation scheme with a max-min representation has the consistency property. The statement of this lemma, however, suggests that its converse is also true, i.e., every smooth consistent method has the max-min representation with a negligible term and so has the monotonicity in an approximation sense, since their max-min representation is approximately monotone. Therefore our task is to justify this observation in our situation.

The present paper is organized as follows. In Section 2, we briefly review the meshfree interpolation theory and derive a general collocation method for (1.1). We rigorously state our assumptions and prove the convergence property in Section 3. Section 4 exhibits a numerical example.

## 2 Generalization of Kansa's method

Throughout this paper, for  $a = (a_i) \in \mathbb{R}^\ell$  and  $\tilde{a} \in \mathbb{R}^{\ell_1 \times \ell_2}$ , we write  $|a| = (\sum_{i=1}^\ell a_i^2)^{1/2}$  and  $|\tilde{a}| = \sup_{y \in \mathbb{R}^{\ell_2} \setminus \{0\}} |\tilde{a}y|/|y|$ , respectively. We denote by  $a^\top$  the transpose of a vector or matrix  $a$ . By  $C$  we denote positive constants that may not be necessarily equal with each other. We also write  $C_{\kappa_1, \dots, \kappa_\ell}$  for a positive constant  $C$  depending only on parameters  $\kappa_1, \dots, \kappa_\ell$ . For a multiindex  $\alpha = (\alpha_1, \dots, \alpha_d)$  of nonnegative integers and a function  $u$ , we define  $D^\alpha u(x)$  by the usual manner, i.e.,

$$D^\alpha u(x) = \frac{\partial^{|\alpha|_1} u(x)}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$$

with  $|\alpha|_1 = \alpha_1 + \cdots + \alpha_d$ . For  $m \in \mathbb{N} \cup \{0\}$  we denote by  $\Pi_m(\mathbb{R}^\ell)$  the set of all  $\mathbb{R}^\ell$ -valued polynomial of degree at most  $m$ .

In this section, we describe a meshfree collocation method for (1.1), which is a generalization of Kansa's method in the parabolic cases. First, we briefly review the basis of the interpolation theory with conditionally positive definite kernels. We refer to Wendland [13] for a complete account. In general, a meshfree method seeks an approximate function in the space spanned by a prespecified kernel. As the kernel we consider a smooth, symmetric conditionally positive definite kernel  $\Phi : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  of order  $m$ . More precisely,  $\Phi$  is assumed to satisfy the following:

- (i)  $\Phi \in C^{2v}(\mathbb{R}^d \times \mathbb{R}^d)$  for some  $v \geq 2$ ;
- (ii)  $\Phi(x, y) = \Phi(y, x)$  for  $x, y \in \mathbb{R}^d$ ;
- (iii) for every  $\ell \in \mathbb{N}$ , for all pairwise distinct  $y_1, \dots, y_\ell \in \mathbb{R}^d$  and for all  $\alpha \in \mathbb{R}^\ell \setminus \{0\}$  satisfying

$$(2.1) \quad \sum_{j=1}^{\ell} \alpha_j \pi(y_j) = 0, \quad \pi \in \Pi_{m-1}(\mathbb{R}^d),$$

we have

$$(2.2) \quad \sum_{i,j=1}^{\ell} \alpha_i \alpha_j \Phi(y_i, y_j) > 0.$$

If (2.2) holds without (2.1), then  $\Phi$  is called a positive definite kernel.

**Example 2.1.** Here are some examples of the conditionally positive definite kernels. In each case,  $\Phi$  is given by  $\Phi(x, y) = \phi(|x - y|)$ , where  $\phi : [0, \infty) \rightarrow \mathbb{R}$ , called a radial basis function (RBF).

- (i) Gaussian RBF:  $\phi(r) = e^{-\alpha r^2}$ ,  $r \geq 0$ , with  $\alpha > 0$ . In this case,  $\Phi$  is positive definite.
- (ii) multiquadric RBF:  $\phi(r) = (\alpha^2 + r^2)^\beta$ ,  $r \geq 0$ , with  $\alpha \in \mathbb{R}$ ,  $\beta \in \mathbb{R} \setminus (\mathbb{N} \cup \{0\})$ . In this case,  $\Phi$  is positive definite for  $\beta < 0$ .

Let  $\Omega$  be a bounded open subset of  $\mathbb{R}^d$ . Suppose that we are in a position to compute a numerical solution of (1.1) on  $\Omega$ . Then assume that  $\Omega$  satisfies an interior cone condition, i.e., there exists  $\theta \in (0, \pi/2)$  and  $r > 0$  such that for any  $x \in \Omega$ ,

$$C(x, \zeta(x), \theta, r) := \left\{ x + \lambda y : y \in \mathbb{R}^d, |y| = 1, y^\top \zeta(x) \geq \cos \theta, \lambda \in [0, r] \right\} \subset \Omega$$

holds for some  $\zeta(x) \in \mathbb{R}^d$  with  $|\zeta(x)| = 1$ .

Let  $X = \{x^{(1)}, \dots, x^{(N)}\}$  be a set of pairwise distinct points in  $\Omega$ . Let  $\pi_1, \dots, \pi_Q$  be a basis of  $\Pi_{m-1}(\mathbb{R}^d)$ , where  $Q = \dim(\Pi_{m-1}(\mathbb{R}^d)) = (m+d)!/(m!d!)$ . Denote  $P = (\pi_k(x^{(j)})) \in \mathbb{R}^{N \times Q}$  and  $A_{\Phi, X} = \{\Phi(x^{(i)}, x^{(j)})\}_{1 \leq i, j \leq N}$ . We assume that  $X$  is a  $\Pi_{m-1}(\mathbb{R}^d)$ -unisolvent set, i.e.,  $\pi \in \Pi_{m-1}(\mathbb{R}^d)$  with  $\pi(x) = 0$  on  $X$  must be zero polynomial. Then, it follows from [13, Theorem 8.21] that the system

$$(2.3) \quad \begin{pmatrix} A_{\Phi, X} & P \\ P^\top & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

has a unique solution  $(\xi(b), \eta(b)) \in \mathbb{R}^N \times \mathbb{R}^Q$  for any  $b \in \mathbb{R}^N$ . Thus, the function

$$I_{g, X}(x) = \sum_{j=1}^N \xi_j(g|_X) \Phi(x, x^{(j)}) + \sum_{i=1}^Q \eta_i(g|_X) \pi_i(x), \quad x \in \Omega,$$

that interpolates  $g$  on  $X$  becomes an approximation of  $g$ . Here,  $\xi(b) = (\xi_1(b), \dots, \xi_N(b))^\top$ ,  $\eta(b) = (\eta_1(b), \dots, \eta_Q(b))^\top$  for  $b \in \mathbb{R}^N$ , and we have set  $g|_X = (g(x^{(1)}), \dots, g(x^{(N)}))^\top$ .

*Remark 2.2.* If  $\Phi$  is positive definite, then the matrix  $A_{\Phi, X}$  is invertible and for  $b \in \mathbb{R}^N$  the solution of (2.3) is given by

$$\xi(b) = A_{\Phi, X}^{-1} b, \quad \eta(b) = 0.$$

In particular, we can drop the polynomial term in the interpolation.

Next we recall the error estimation results for interpolation by conditionally positive definite kernels. Let  $\mathcal{N}_\Phi(\Omega)$  be the native space corresponding to  $\Phi$ . See [13] for a precise definition. Here, we remark that  $\mathcal{N}_\Phi(\Omega)$  is a linear subspace of  $C(\Omega)$  equipped with a semi-inner product  $(\cdot, \cdot)_{\mathcal{N}_\Phi(\Omega)}$ . If  $g, g' \in C(\Omega)$  are of the form

$$g(x) = \sum_{j=1}^M \alpha_j \Phi(x, y_j), \quad g'(x) = \sum_{j=1}^M \alpha'_j \Phi(x, y'_j), \quad x \in \Omega,$$

where  $M, M' \in \mathbb{N}$ ,  $\alpha, \alpha' \in \mathbb{R}^N$ ,  $y_1, \dots, y_M, y'_1, \dots, y'_{M'} \in \Omega$ , with  $\sum_{j=1}^M \alpha_j \pi(y_j) = \sum_{j=1}^{M'} \alpha'_j \pi(y'_j) = 0$  for all  $\pi \in \Pi_{m-1}(\mathbb{R}^d)$ , then

$$(g, g')_{\mathcal{N}_\Phi(\Omega)} = \sum_{j=1}^M \sum_{\ell=1}^{M'} \alpha_j \alpha'_\ell \Phi(y_j, y'_\ell).$$

**Example 2.3.** Suppose that  $\Phi$  is given by  $\Phi(x, y) = \phi(|x - y|)$  where  $\phi$  is some function on  $[0, \infty)$  such that  $x \mapsto \phi(|x|)$  is integrable and has a Fourier transform that decays as  $(1 + |\cdot|^2)^{-k}$ ,  $k \in \mathbb{N}$ ,  $k > d/2$ . Suppose moreover that  $\Omega$  has a Lipschitz boundary. Then  $\mathcal{N}_\Phi(\Omega)$  coincides with the  $L^2$ -Sobolev space on  $\Omega$  of order  $k$  with equivalent norms.

The error of the interpolation is estimated as follows: for every  $g \in \mathcal{N}_\Phi(\Omega)$  and every multi-index  $\alpha$  with  $|\alpha| \leq \nu$ ,

$$(2.4) \quad |D^\alpha g(x) - D^\alpha I_{g,X}(x)| \leq C_{\nu,\Phi} \Delta_{X,\Omega}^{\nu-|\alpha|} |g|_{\mathcal{N}_\Phi(\Omega)}, \quad x \in \Omega,$$

where  $|\cdot|_{\mathcal{N}_\Phi(\Omega)} = (\cdot, \cdot)_{\mathcal{N}_\Phi(\Omega)}^{1/2}$  and  $\Delta_{\Omega,X}$  is the fill distance defined by

$$\Delta_{\Omega,X} = \sup_{x \in \Omega} \min_{j=1,\dots,N} |x - x^{(j)}|.$$

In the above, we have assumed that  $\Omega$  satisfies an interior cone condition and  $X$  is  $\Pi_{m-1}(\mathbb{R}^d)$ -unisolvent. Typical examples are the cases that  $\Omega$  is an ball or rectangular and  $X$  is a set of the equi-spaced grid points. These would be known facts, but for reader's convenience, we give proofs.

**Proposition 2.4.** *We have the following:*

- (i) *Let  $x_0 \in \mathbb{R}^d$  and  $R > 0$  be given. Then the set  $\{x \in \mathbb{R}^d : |x - x_0| < R\}$  satisfies an interior cone condition.*
- (ii) *Let  $a_\ell, b_\ell \in \mathbb{R}$  with  $a_\ell < b_\ell$ ,  $\ell = 1, \dots, d$ , be given. Then the set  $\prod_{\ell=1}^d (a_\ell, b_\ell)$  satisfies an interior cone condition.*
- (iii) *Suppose that  $N \geq m$  and that for any  $\ell = 1, \dots, d$ , the sequence of  $\ell$ -th coordinate  $x_\ell^{(1)}, \dots, x_\ell^{(N)}$  is pairwise distinct. Then  $X$  is  $\Pi_{m-1}(\mathbb{R}^d)$ -unisolvent.*

*Proof.* To prove (i), set  $r = R/(1 + \sqrt{2})$  and  $\theta = \pi/4$  and denote by  $\Omega_1$  the open ball. For  $x_0 \in \Omega_1$ , any choice for  $\zeta(x_0)$  yields  $C(x_0, \zeta(x_0), \theta, r) \subset \Omega$ . For  $x \in \Omega_1 \setminus \{x_0\}$ , define  $\zeta(x) = -(x - x_0)/|x - x_0|$ . Let  $\lambda \in [0, r]$  and  $y \in \mathbb{R}^d$  with  $|y| = 1$  and  $y^\top \zeta(x) \geq \cos \theta$ . If  $|x - x_0| < \sqrt{2}r$ , then

$$|x + \lambda y - x_0| \leq |x - x_0| + \lambda |y| < \sqrt{2}r + r = R.$$

If  $|x - x_0| \geq \sqrt{2}r$ , then

$$|x + \lambda y - x_0|^2 = |x - x_0|^2 + 2\lambda y^\top (x - x_0) + \lambda^2 \leq |x - x_0|^2 - \sqrt{2}|x - x_0|\lambda + \lambda^2.$$

The quadratic function of  $\lambda$  on the right-hand side in the above inequality attains its maximum over  $[0, r]$  at  $\lambda = 0$ . Thus  $|x + \lambda y - x_0| < R$ . Therefore we obtain  $C(x, \zeta(x), \theta, r) \subset \Omega_1$ .

We turn to the next assertion. The case  $d = 1$  is easy to show, thus we assume  $d \geq 2$ . As a preliminary, we first confirm the following simple fact: for  $y = (y_1, \dots, y_d)^\top \in \mathbb{R}^d$ ,

$$(2.5) \quad \sum_{i=1}^d y_i^2 = 1, \quad \sum_{i=1}^d y_i \geq \sqrt{d-1} \quad \implies \quad y_i \geq 0, \quad i = 1, \dots, d.$$

To see this, let  $k \in \{1, \dots, d\}$  be arbitrary. Then by Cauchy-Schwartz inequality,

$$\sqrt{d-1} - y_k \leq \sum_{i \neq k} y_i \leq \sqrt{d-1} \left( \sum_{i \neq k} y_i^2 \right)^{1/2} = \sqrt{(d-1)(1-y_k^2)}.$$

Since  $|y_k| \leq 1$ , we have  $\sqrt{d-1} - y_k \geq 0$ . Thus the inequality above is equivalent to  $d-1-2\sqrt{d-1}y_k+y_k^2 \leq (d-1)(1-y_k^2)$  and this is also equivalent to  $y_k(dy_k-2\sqrt{d-1}) \leq 0$ . Hence we find that  $y_k \geq 0$ .

Now, set  $r = \min_i(b_i - a_i)/4$  and  $\theta \in (0, \pi/2)$  such that  $\cos \theta = \sqrt{(d-1)/d}$  and denote by  $\Omega_2$  the open rectangular set. Fix  $x = (x_1, \dots, x_d)^\top \in \Omega_2$ . Define  $\zeta(x) = (\zeta(x)_1, \dots, \zeta(x)_d)^\top$  by  $\zeta_i(x) = 1$  if  $x_i \leq (a_i + b_i)/2$ ,  $= -1$  otherwise. Moreover define

$$I_+(x) = \{k \in \{1, \dots, d\} : \zeta_k(x) > 0\}, \quad I_-(x) = \{k \in \{1, \dots, d\} : \zeta_k(x) < 0\}.$$

Suppose that  $x + \lambda y \in C(x, \zeta(x), \theta, r)$ . Then,

$$\sum_{k \in I_-(x)} (-y_k)^2 + \sum_{k \in I_+(x)} y_k^2 = 1, \quad \sum_{k \in I_-(x)} (-y_k) + \sum_{k \in I_+(x)} y_k \geq \sqrt{d-1}.$$

It follows from (2.5) that  $y_k \geq 0$  for  $k \in I_+(x)$  and  $y_k \leq 0$  for  $k \in I_-(x)$ . Thus, for the case  $k \in I_+(x)$  we see  $a_k < x_k \leq x_k + \lambda y_k \leq (a_k + b_k)/2 + (b_k - a_k)/2 < b_k$  for  $\lambda \in [0, r]$ . For the case  $k \in I_-(x)$ , we have  $x_k + \lambda y_k \leq x_k < b_k$  and  $x_k + \lambda y_k > (a_k + b_k)/2 - r \geq (a_k + b_k)/2 - (b_k - a_k)/2 = a_k$ . Consequently, we deduce that  $C(x, \zeta(x), \theta, r) \subset \Omega_2$ .

Finally, we prove (iii) by the induction on  $d$ . To this end, define

$$X_\ell = \{(x_1^{(1)}, \dots, x_\ell^{(1)})^\top, \dots, (x_1^{(N)}, \dots, x_\ell^{(N)})^\top\}, \quad \ell = 1, \dots, d.$$

Let  $\pi \in \Pi_{m-1}(\mathbb{R})$  and represent this as  $\pi(x) = \sum_{j=0}^{m-1} c_j x^j$ ,  $x \in \mathbb{R}$ . Then,  $\pi = 0$  on  $X_1$  is equivalent to

$$\begin{pmatrix} 1 & x_1^{(1)} & \cdots & (x_1^{(1)})^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_1^{(N)} & \cdots & (x_1^{(N)})^{m-1} \end{pmatrix} \begin{pmatrix} c_0 \\ \vdots \\ c_{m-1} \end{pmatrix} = 0.$$

Since  $x_1^{(j)}$ 's are pairwise distinct, the Vandermonde matrix in the above equality has the full rank. This and  $N \geq m$  means  $c_0 = \dots = c_{m-1} = 0$ . Thus  $\pi = 0$  on  $\mathbb{R}$ .

Now assume that  $X_\ell$  is  $\Pi_{m-1}(\mathbb{R}^\ell)$ -unisolvent. Let  $\pi \in \Pi_{m-1}(\mathbb{R}^{\ell+1})$  and represent this as  $\pi(x) = \sum_{j=0}^{m-1} \pi_j(\tilde{x})(x_{\ell+1})^j$  for  $x = (x_1, \dots, x_{\ell+1})^\top$ , where  $\pi_j \in \Pi_{m-1-j}(\mathbb{R}^\ell)$ ,  $j = 0, \dots, m-1$ , and  $\tilde{x} = (x_1, \dots, x_\ell)^\top$ . Suppose that  $\pi = 0$  on  $X_{\ell+1}$ . Then we have

$$\begin{pmatrix} 1 & x_{\ell+1}^{(1)} & \cdots & (x_{\ell+1}^{(1)})^{m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{\ell+1}^{(N)} & \cdots & (x_{\ell+1}^{(N)})^{m-1} \end{pmatrix} \begin{pmatrix} \pi_0(\tilde{x}) \\ \vdots \\ \pi_{m-1}(\tilde{x}) \end{pmatrix} = 0$$

for any  $\tilde{x} \in X_\ell$ . As in the above arguments, we deduce that  $\pi_j(\tilde{x}) = 0$ ,  $j = 0, \dots, m-1$ , for any  $\tilde{x} \in X_\ell$ . Thus by the assumption of the induction, we obtain  $\pi_0 = \dots = \pi_{m-1} = 0$  on  $\mathbb{R}^\ell$ . This means  $\pi = 0$  on  $\mathbb{R}^{\ell+1}$ .  $\square$

Now, let us describe the meshfree collocation methods for our parabolic equations. We start with the formal time discretization of (1.1) to get

$$(2.6) \quad \frac{v(t_{k+1}, x) - v(t_k, x)}{h} \simeq \theta F(t_{k+1}, x; v(t_{k+1}, \cdot)) + (1 - \theta) F(t_k, x; v(t_k, \cdot))$$

where  $t_k = kh$ ,  $k = 0, \dots, n$  and  $h = T/n$ ,  $\theta \in [0, 1]$ , and for any  $\varphi \in C^2(\mathbb{R}^d)$

$$F(t, x; \varphi) = F(t, x, \varphi(x), D\varphi(x), D^2\varphi(x)), \quad x \in \mathbb{R}^d.$$

Let us denote by  $v_{k,j}$ ,  $k = 0, \dots, n$ ,  $j = 1, \dots, N$ , an approximate solution of (1.1) at  $\{t_0, \dots, t_n\} \times X$ , to be determined below, and set  $v^h(t_k, \cdot)$  by the meshfree interpolation of  $\{v_{k,j}\}_{j=1, \dots, N}$ , i.e.,

$$(2.7) \quad v^h(t_k, x) = \sum_{j=1}^N \xi_j(v_k) \Phi(x, x^{(j)}) + \sum_{\ell=1}^Q \eta_\ell(v_k) \pi_\ell(x), \quad x \in \Omega,$$

where  $v_k = (v_{k,1}, \dots, v_{k,N})^\top$ . Moreover, assume that  $v^h$  satisfies (2.6) with equality on  $X$ . Then,

$$v_{k+1,j} - v_{k,j} = h\theta \tilde{F}_{k+1,j}(v_{k+1}) + h(1 - \theta) \tilde{F}_{k,j}(v_k), \quad k = 0, \dots, n-1, \quad j = 1, \dots, N.$$

Here,  $\tilde{F}_{k,j}(v_k) = F(t_k, x^{(j)}; v^h(t_k, \cdot))$ . Thus, denoting  $\tilde{F}_k(v_k) = (\tilde{F}_{k,1}(v_k), \dots, \tilde{F}_{k,N}(v_k))^\top$ , we get

$$(2.8) \quad v_k + h(1 - \theta) \tilde{F}_k(v_k) = v_{k+1} - h\theta \tilde{F}_{k+1}(v_{k+1}), \quad k = 0, \dots, n-1.$$

The terminal condition  $v^h(t_n, \cdot)$  is given by

$$(2.9) \quad v^h(t_n, x) = I_{f,X}(x), \quad x \in \Omega.$$

Consequently, our method is described as follows: determine values of grid points  $\{t_0, \dots, t_n\} \times X$  by solving the equation (2.8) with (2.9). Then define the function  $v^h$  on  $\{t_0, \dots, t_n\} \times \Omega$  by (2.7), which is a candidate of an approximate solution of (1.1).

*Remark 2.5.* The linearity of  $(\xi(b), \eta(b))$  with respect to  $b$  yields

$$(2.10) \quad v^h(t_k, x) = v^h(t_{k+1}, x) - h(1 - \theta) I_{F(t_k, \cdot; v^h(t_k, \cdot)), X}(x) - h\theta I_{F(t_{k+1}, \cdot; v^h(t_{k+1}, \cdot)), X}(x), \quad x \in \Omega.$$

In the case of  $\theta = 1$ , the equation (2.8) becomes a simple recursion formula, and then  $v^h$  is computed by the repeated interpolation procedures, i.e.,

$$v^h(t_k, x) = v^h(t_{k+1}, x) - h I_{F(t_{k+1}, \cdot; v^h(t_{k+1}, \cdot)), X}(x), \quad x \in \Omega.$$

### 3 Convergence

We consider the terminal value problem (1.1) under the following assumptions:

**Assumption 3.1.** (i) For  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $z \in \mathbb{R}$ ,  $p \in \mathbb{R}^d$ , and  $\Gamma, \Gamma' \in \mathbb{S}^d$  with  $\Gamma \geq \Gamma'$ ,

$$F(t, x, z, p, \Gamma) \leq F(t, x, z, p, \Gamma').$$

(ii) There exist a continuous function  $F_0$  on  $[0, T] \times \mathbb{R}^d \times \mathbb{R}$  and a constant  $K_0 \in (0, \infty)$  such that

$$|F(t, x, z, p, \Gamma) - F(t', x', z', p', \Gamma)| \leq |F_0(t, x, z) - F_0(t', x', z')| + K_0(|p - p'| + |\Gamma - \Gamma'|)$$

for  $t, t' \in [0, T]$ ,  $x, x' \in \mathbb{R}^d$ ,  $z, z' \in \mathbb{R}$ ,  $p, p' \in \mathbb{R}^d$ , and  $\Gamma, \Gamma' \in \mathbb{S}^d$ .

(iii) There exists a constant  $K_1 \in (0, \infty)$  such that

$$|F(t, x, z, p, \Gamma)| \leq K_1(1 + |z| + |p| + |\Gamma|)$$

for  $t \in [0, T]$ ,  $x \in \mathbb{R}^d$ ,  $z \in \mathbb{R}$ ,  $p \in \mathbb{R}^d$ , and  $\Gamma \in \mathbb{S}^d$ .

(iv) The function  $f$  is continuous and bounded on  $\mathbb{R}^d$ .

Recall that an  $\mathbb{R}$ -valued, upper-semicontinuous function  $u$  on  $[0, T] \times \mathbb{R}^d$  is said to be a viscosity subsolution of (1.1) if the following two conditions hold:

(i) for every  $(t, x) \in [0, T] \times \mathbb{R}^d$  and every smooth function  $\varphi$  such that  $u - \varphi$  has a local maximum at  $(t, x)$  we have

$$-\partial_t \varphi(t, x) + F(t, x, u(t, x), D\varphi(t, x), D^2 \varphi(t, x)) \leq 0;$$

(ii)  $u(T, x) \leq f(x)$ ,  $x \in \mathbb{R}^d$ .

Similarly, an  $\mathbb{R}$ -valued, lower-semicontinuous function  $u$  on  $[0, T] \times \mathbb{R}^d$  is said to be a viscosity supersolution of (1.1) if the following two conditions hold:

(i) for every  $(t, x) \in [0, T] \times \mathbb{R}^d$  and every smooth function  $\varphi$  such that  $u - \varphi$  has a local minimum at  $(t, x)$  we have

$$-\partial_t \varphi(t, x) + F(t, x, u(t, x), D\varphi(t, x), D^2 \varphi(t, x)) \geq 0;$$

(ii)  $u(x) \geq f(x)$ ,  $x \in \mathbb{R}^d$ .

We say that  $u$  is a viscosity solution of (1.1) if it is both a viscosity subsolution and a viscosity supersolution of (1.1).

We assume that the following comparison principle holds:

**Assumption 3.2.** For every bounded, upper-semicontinuous viscosity subsolution  $u$  of (1.1) and bounded lower-semicontinuous viscosity supersolution  $w$  of (1.1), we have

$$u(t, x) \leq w(t, x), \quad (t, x) \in [0, T] \times \mathbb{R}^d.$$

Under Assumptions 3.1 and 3.2, there exists a unique continuous viscosity solution  $v$  of (1.1). See [10].

**Assumption 3.3.** The equation (2.8) has a unique solution  $v_{k,j}$ ,  $k = 0, \dots, n-1$ ,  $j = 1, \dots, N$ .

Notice that Assumption 3.3 is trivially satisfied when  $\theta = 1$ .

Set  $L_N = |A_{\Phi, X}^{-1}|$  if  $\Phi$  is simply positive definite and  $L_N = |\tilde{A}_{\Phi, X}^{-1}|$  if  $\Phi$  is conditionally positive definite of order  $m \geq 1$ , where  $\tilde{A}_{\Phi, X}$  denotes the matrix on the left-hand side in (2.3).

Hereafter, we assume that the number  $N$  of data sites is a function of the time step  $h$ . To control the bound of  $v^h$ , we make the following assumptions:



**Assumption 3.4.** The function  $L_N$  of  $h$  is bounded away from zero and there exists  $K_3 \in (0, \infty)$ ,  $\delta \in (0, 1/5)$ ,  $h_0 \in (0, 1)$  such that

$$h^\delta \sqrt{N} L_N \exp(\sqrt{3} T K_1 K_2 (1 + \sqrt{N}) L_N) \leq K_3, \quad h \leq h_0,$$

where

$$K_2 = \max \left\{ \left( \sum_{|\alpha| \leq 3} \max_{x, y \in \Omega} |D^\alpha \Phi(x, y)|^2 \right)^{1/2}, \left( \sum_{|\alpha| \leq 3} \max_{x \in \Omega} \sum_{\ell=1}^Q |D^\alpha \pi_\ell(x)|^2 \right)^{1/2} \right\}.$$

Here,  $D^\alpha \Phi(x, y)$  is interpreted as the partial derivative of  $\Phi$  with respect to the first argument.

To discuss Assumption 3.4, recall that the set  $X$  of data sites is said to be quasi-uniform with respect to a constant  $c_{qu} > 0$  if

$$q_X \leq \Delta_{\Omega, X} \leq c_{qu} q_X,$$

where  $q_X$  is the separation distance of  $X$ , defined by

$$q_X = \frac{1}{2} \min_{i \neq j} |x^{(i)} - x^{(j)}|.$$

A typical example of quasi-uniform data sites is, of course, a set of equi-spaced grid points. It is known that if  $X$  is quasi-uniform with respect to  $c_{qu} > 0$ , then there exists constants  $c_1, c_2 > 0$ , only depending on  $d$  and  $c_{qu}$ , such that

$$c_1 N^{-1/d} \leq q_X \leq c_2 N^{-1/d}.$$

**Example 3.5.** (i) In the case of  $\Phi(x, y) = e^{-\alpha|x-y|^2}$ ,  $\alpha > 0$ , it is known that

$$|A_{\Phi, X}^{-1}| \leq \frac{(2\alpha)^{d/2}}{\tilde{c}_{d,1}} q_X^d e^{40.71 d^2 / (\alpha q_X^2)}$$

where

$$\tilde{c}_{d,1} = \frac{1}{2\Gamma((d+2)/2)} \left( \frac{\tilde{c}_{d,2}}{\sqrt{8}} \right)^d, \quad \tilde{c}_{d,2} = 12 \left( \frac{\pi \Gamma^2((d+2)/2)}{9} \right)^{1/(d+1)},$$

and  $\Gamma$  denotes the Gamma function (see [13, Chapter 12]). Thus, if  $X$  is quasi-uniform, then

$$L_N = |A_{\Phi, X}^{-1}| \leq \frac{(2\alpha)^{d/2}}{\tilde{c}_{d,1}} c_2^2 N^{-1} e^{40.71 d^2 N^{2/d} / (\alpha c_1^2)}.$$

(ii) In the case of  $\Phi(x, y) = (\alpha^2 + |x-y|^2)^{-\beta}$ ,  $\alpha, \beta > 0$ , it is known that

$$|A_{\Phi, X}^{-1}| \leq \tilde{c}_{d, \alpha, \beta} q_X^{\beta+d/2-1/2} \exp(2\alpha \tilde{c}_{d,2} / q_X)$$

with an explicitly known constant  $\tilde{c}_{d, \alpha, \beta}$  (see [13, Chapter 12]). Thus, if  $X$  is quasi-uniform, then

$$L_N = |A_{\Phi, X}^{-1}| \leq \tilde{c}_{d, \alpha, \beta} c_2^{\beta+d/2-1/2} N^{-(\beta+d/2-1/2)/d} e^{2\alpha \tilde{c}_{d,2} N^{1/d} / c_1}.$$

To ensure the convergence of the interpolation at each time step, we impose the following conditions in view of (2.4):

**Assumption 3.6.** (i) The terminal data  $f$  and the function  $F(t_k, \cdot; v^h(t_k, \cdot))$  belong to  $\mathcal{N}_\Phi(\Omega)$  for every  $k = 0, \dots, n-1$ .

(ii) The meshfree approximation at each time step is successful, i.e.,

$$\Delta_{X,\Omega}^v \left( 1 + \max_{k=0,\dots,n-1} |F(t_k, \cdot; v^h(t_k, \cdot))|_{\mathcal{N}_\Phi(\Omega)} \right) \rightarrow 0, \quad h \rightarrow 0.$$

To prove the convergence, we define  $v^h(s, x)$  for  $s \in (t_k, t_{k+1})$  by any continuous interpolation of  $v^h(t_k, x)$  and  $v^h(t_{k+1}, x)$ ,  $k = 0, \dots, n-1$ .

**Theorem 3.7.** *Suppose that Assumptions 3.1-3.4, 3.6 hold. Then we have*

$$\lim_{h \searrow 0} \sup_{s \rightarrow t} \sup_{x \in \Omega} |v^h(s, x) - v(t, x)| = 0.$$

*Remark 3.8.* In the theorem above, in addition to Assumptions 3.1-3.4 and 3.6, we have assumed  $\Omega$  to be a bounded open subset of  $\mathbb{R}^d$  and to satisfy an interior cone condition, and  $X$  to be  $\Pi_{m-1}(\mathbb{R}^d)$ -unisolvent. All these conditions are satisfied when  $\Omega$  is an open rectangular set and  $X$  is a set of equi-spaced grid points of  $\Omega$  with  $N \geq m$ .

To prove Theorem 3.7, we need some preliminaries.

**Lemma 3.9.** *Suppose that Assumptions 3.1-3.4 hold. Then, there exist  $h_1 \in (0, 1)$  such that for  $h \leq h_1$*

$$\max_{k=0,\dots,n-1} |v_k| \leq \left( \sup_{x \in \Omega} |f(x)| + \frac{1}{\sqrt{2}K_2} \right) \exp(\sqrt{3}TK_1K_2\sqrt{N}(1+\sqrt{N})L_N).$$

*Proof.* Since  $|\xi(v_k)| + |\eta(v_k)| \leq \sqrt{2}L_N|v_k|$  for any  $k$ , we have

$$\sum_{i=0}^2 |D^i v^h(t_k, x)| \leq \sqrt{2}K_2(1+\sqrt{N})L_N|v_k|.$$

This and Assumption 3.1 imply

$$|\tilde{F}_{k,j}(v_k)| \leq K_1 + K_1 \sum_{i=0}^2 \left| D^i v^h(t_k, x^{(j)}) \right| \leq K_1 + \sqrt{2}K_1K_2(1+\sqrt{N})L_N|v_k|.$$

Using  $|y| \leq \sqrt{N} \max_{j=1,\dots,N} |y_j|$  for  $y = (y_1, \dots, y_N)^\top \in \mathbb{R}^N$ , we find that

$$|\tilde{F}_k(v_k)| \leq K_1\sqrt{N} + \sqrt{2}K_1K_2\sqrt{N}L_N|v_k| + \sqrt{2}K_1K_2NL_N|v_k|.$$

Hence,

$$\begin{aligned} |v_k| &\leq |v_{k+1}| + h(1-\theta)|\tilde{F}_k(v_k)| + h\theta|\tilde{F}_{k+1}(v_{k+1})| \\ &\leq \left( 1 + \sqrt{2}h\theta K_1K_2\sqrt{N}(1+\sqrt{N})L_N \right) |v_{k+1}| + \sqrt{2}h(1-\theta)K_1K_2\sqrt{N}(1+\sqrt{N})L_N|v_k| + hK_1\sqrt{N}. \end{aligned}$$

Assumption 3.4 implies

$$h^\delta \sqrt{N} L_N (1 + \sqrt{N}) \leq h^\delta \sqrt{N} L_N \frac{L_N}{C_0} (1 + \sqrt{N}) \leq h^\delta \sqrt{N} L_N \frac{\exp(\sqrt{3} T K_1 K_2 (1 + \sqrt{N}) L_N)}{\sqrt{3} C_0 T K_1 K_2} \leq C$$

for  $h \leq h_0$ , where  $C_0$  is a lower bound for  $L_N$ . Thus,

$$\sqrt{2} h (1 - \theta) K_1 K_2 \sqrt{N} L_N (1 + \sqrt{N}) \leq 1 - \frac{\sqrt{2}}{\sqrt{3}} < 1, \quad h \leq h_1$$

for some  $h_1 \leq h_0$ , it follows that for any  $k = 0, \dots, n-1$ ,  $h \leq h_1$ ,

$$\begin{aligned} |v_k| &\leq \frac{1 + \sqrt{2} h \theta K_1 K_2 \sqrt{N} (1 + \sqrt{N}) L_N}{1 - \sqrt{2} h (1 - \theta) K_1 K_2 \sqrt{N} (1 + \sqrt{N}) L_N} |v_{k+1}| + \frac{h K_1 \sqrt{N}}{1 - \sqrt{2} h (1 - \theta) K_1 K_2 \sqrt{N} (1 + \sqrt{N}) L_N} \\ &= \left( 1 + \frac{\sqrt{2} h K_1 K_2 \sqrt{N} (1 + \sqrt{N}) L_N}{1 - \sqrt{2} h (1 - \theta) K_1 K_2 \sqrt{N} (1 + \sqrt{N}) L_N} \right) |v_{k+1}| + \frac{h K_1 \sqrt{N}}{1 - \sqrt{2} h (1 - \theta) K_1 K_2 \sqrt{N} (1 + \sqrt{N}) L_N} \\ &\leq (1 + \sqrt{3} h K_1 K_2 \sqrt{N} (1 + \sqrt{N} L_N)) |v_{k+1}| + \sqrt{3/2} h K_1 \sqrt{N}. \end{aligned}$$

Therefore, we have, for any  $k$

$$\begin{aligned} |v_k| &\leq (1 + \sqrt{3} h K_1 K_2 \sqrt{N} (1 + \sqrt{N}) L_N)^n \sup_{x \in \Omega} |f(x)| \\ &\quad + \sqrt{3/2} h K_1 \sqrt{N} \times \frac{(1 + \sqrt{3} h K_1 K_2 \sqrt{N} (1 + \sqrt{N}) L_N)^n - 1}{\sqrt{3} h K_1 K_2 \sqrt{N} (1 + \sqrt{N}) L_N} \\ &\leq \exp(\sqrt{3} T K_1 K_2 \sqrt{N} (1 + \sqrt{N}) L_N) \sup_{x \in \Omega} |f(x)| \\ &\quad + \frac{1}{\sqrt{2} K_2 (1 + \sqrt{N}) L_N} (\exp(\sqrt{3} T K_1 K_2 \sqrt{N} (1 + \sqrt{N}) L_N) - 1), \end{aligned}$$

leading to the conclusion of the lemma.  $\square$

**Lemma 3.10.** *Suppose that Assumptions 3.1, 3.3 and 3.4 hold. Then there exist a constant  $K_4 \in (0, \infty)$ ,  $h_2 \in (0, 1]$  such that for  $h \leq h_2$  we have the following:*

- (i)  $\sum_{|\alpha|_1 \leq 3} |D^\alpha v^h(t_k, x)| \leq K_4 h^{-\delta}$  for  $k = 0, \dots, n-1$  and  $x \in \Omega$ .
- (ii)  $\sum_{|\alpha|_1 \leq 3} |D^\alpha v^h(t_{k+1}, x) - D^\alpha v^h(t_k, x)| \leq K_4 h^{1-2\delta}$  for  $k = 0, \dots, n-2$  and  $x \in \Omega$ .

*Proof.* Fix  $k = 0, \dots, n-1$  and let  $h_1$  be as in Assumption 3.4. Using the previous lemma, we observe

$$\begin{aligned} \sum_{|\alpha|_1 \leq 3} |D^\alpha v^h(t_k, x)| &\leq \sqrt{2} K_2 (1 + \sqrt{N}) L_N |v_k| \\ &\leq 2 \left( 1 + \sqrt{2} K_2 \sup_{x \in \Omega} |f(x)| \right) \sqrt{N} L_N \exp(\sqrt{3} T K_1 K_2 \sqrt{N} (1 + \sqrt{N}) L_N) \\ &\leq 2 \left( 1 + \sqrt{2} K_2 \sup_{x \in \Omega} |f(x)| \right) K_3 h^{-\delta} \end{aligned}$$

for  $h \leq h_1$ . Thus the first assertion follows.

Next, since  $\xi(b)$  and  $\eta(b)$  is linear in  $b$ , we obtain

$$\begin{aligned} \sum_{|\alpha|_1 \leq 3} |D^\alpha v^h(t_{k+1}, x) - D^\alpha v^h(t_k, x)| &\leq \sqrt{2}K_2\sqrt{N}|\xi(v_{k+1}) - \xi(v_k)| + \sqrt{2}K_2|\eta(v_{k+1}) - \eta(v_k)| \\ &\leq \sqrt{2}K_2(1 + \sqrt{N})L_N|v_{k+1} - v_k|. \end{aligned}$$

Using Assumption 3.1 and the first assertion in this lemma, we see

$$\begin{aligned} |\tilde{F}_k(v_k)| &\leq \left( \sum_{j=1}^N K_1^2 \left( 1 + \sum_{i=0}^2 |D^i v^h(t_k, x^{(j)})| \right)^2 \right)^{1/2} \leq \sqrt{N}K_1 \left( 1 + \sum_{i=0}^2 \sup_{x \in \Omega} |D^i v^h(t_k, x)| \right) \\ &\leq C\sqrt{N}h^{-\delta} \end{aligned}$$

for  $h \leq h_1$ . Hence,

$$|v_{k+1} - v_k| \leq h|\tilde{F}_k(v_k)| + h|\tilde{F}_{k+1}(v_{k+1})| \leq C\sqrt{N}h^{1-\delta}.$$

Therefore, in view of Assumption 3.4,

$$\sum_{|\alpha|_1 \leq 3} |D^\alpha v^h(t_{k+1}, x) - D^\alpha v^h(t_k, x)| \leq C(1 + \sqrt{N})\sqrt{N}L_N h^{1-\delta} \leq Ch^{1-2\delta}$$

for sufficiently small  $h$ . Thus the second assertion follows.  $\square$

Let  $K_4$  as in the previous lemma. For  $h > 0$  and  $\kappa > 0$  define

$$\mathcal{D}_{h,\delta} = \left\{ (p, \Gamma) \in \mathbb{R}^d \times \mathbb{S}^d : |p|, |\Gamma| \leq K_4 h^{-\delta} \right\}, \quad \mathcal{X}_{h,\kappa} = \left\{ w \in \mathbb{R}^d : |w| \leq h^{-\kappa} \right\}.$$

The following lemma is a variant of Lemma 4.1 in [10].

**Lemma 3.11.** *Suppose that Assumption 3.1 holds. Let  $O \subset \mathbb{R}^d$  be open and bounded. Then there exist  $h_3 \in (0, 1]$ ,  $\beta \in (0, \infty)$ ,  $\kappa \in (0, \infty)$  such that for  $(t, x, z) \in [0, T] \times O \times \mathbb{R}$ ,  $\{\varphi^h\}_{h \in (0, h_3]} \subset C^3(O)$  with  $\sum_{|\alpha|_1 \leq 3} \sup_{y \in O} |D^\alpha \varphi^h(y)| \leq K_4 h^{-\delta}$ , and  $h \in (0, h_3]$ ,*

$$\begin{aligned} &\left| \varphi^h(x) - hF(t, x, z, D\varphi^h(x), D^2\varphi^h(x)) \right. \\ &\quad \left. - \sup_{(p, \Gamma) \in \mathcal{D}_{h,\delta}} \inf_{w \in \mathcal{X}_{h,\kappa}} \left[ \varphi^h(x + \sqrt{h}w) - \sqrt{h}w^\top p - \frac{h}{2}w^\top \Gamma w - hF(t, x, z, p, \Gamma) \right] \right| \leq C_{K_0, K_4} h^{1+\beta}. \end{aligned}$$

*Proof.* First, fix arbitrary  $\varphi^h \in C^3(O)$  with  $\sum_{|\alpha|_1 \leq 3} \sup_{y \in O} |D^\alpha \varphi^h(y)| \leq K_4 h^{-\delta}$  and  $(t, x, z) \in [0, T] \times O \times \mathbb{R}$ . Then set  $\varphi = \varphi^h$  and  $p_0 = D\varphi(x)$ ,  $\Gamma_0 = D^2\varphi(x)$ . Also, for simplicity, we write  $F(p, \Gamma) = F(t, x, z, p, \Gamma)$  for  $(p, \Gamma) \in \mathcal{D}_{h,\delta}$ . Since  $\delta < 1/5$ , there exists  $\varepsilon > 0$  such that  $\delta < 1/(5 + \varepsilon)$ . Then define  $\kappa > 0$  by

$$\kappa = \frac{1}{3} \left( \frac{5}{10 + 2\varepsilon} - \delta \right).$$

Next, take  $h_3 \in (0, 1]$  such that  $x + \sqrt{h}w \in \Omega$  for all  $w \in \mathcal{X}_{h,\kappa}$ , and  $h \in (0, h_3]$ . By Taylor expansion of  $\varphi$  up to the second term, we have

$$\begin{aligned} & \sup_{(p,\Gamma) \in \mathcal{D}_{h,\delta}} \inf_{w \in \mathcal{X}_{h,\kappa}} \left[ \varphi(x + \sqrt{h}w) - \sqrt{h}w^\top p - \frac{h}{2}w^\top \Gamma w - hF(p, \Gamma) \right] \\ & \geq \varphi(x) - Ch^{-\delta+3/2-3\kappa} + \sup_{(p,\Gamma) \in \mathcal{D}_{h,\delta}} \inf_{w \in \mathcal{X}_{h,\kappa}} \left[ \sqrt{h}w^\top (p_0 - p) + \frac{h}{2}w^\top (\Gamma_0 - \Gamma)w - hF(p, \Gamma) \right]. \end{aligned}$$

Then, considering  $p = p_0$  and  $\Gamma = \Gamma_0$ , we find that the right-hand side in the above inequality is greater than  $\varphi(x) - Ch^{1+\varepsilon/(10+2\varepsilon)} - hF(p_0, \Gamma_0)$ .

To show the reverse inequality, let  $(p, \Gamma) \in \mathcal{D}_{h,\delta}$ . Since  $2\kappa - \delta = (5/3)(1/(5+\varepsilon) - \delta) > 0$ , we can take  $\gamma \in (0, 2\kappa - \delta)$ . Suppose that the minimum eigenvalue of  $\Gamma_0 - \Gamma$  is greater than or equal to  $-h^\gamma$ . Then  $\Gamma \leq \Gamma_0 + h^\gamma I$  so that

$$F(p, \Gamma) \geq F(p, \Gamma_0 + h^\gamma I) \geq F(p_0, \Gamma_0) - K_0|p - p_0| - K_0h^\gamma.$$

The last inequality follows from Assumption 3.1 (ii), i.e., the Lipschitz continuity of  $F(p, \Gamma)$ . Thus,

$$\begin{aligned} (3.1) \quad & \sqrt{h}(p_0 - p)^\top w + \frac{h}{2}w^\top (\Gamma_0 - \Gamma)w - hF(p, \Gamma) \\ & \leq \sqrt{h}(p_0 - p)^\top w + K_4h^{1-\delta}|w|^2 - hF(p_0, \Gamma_0) + K_0h|p - p_0| + K_0h^{1+\gamma}. \end{aligned}$$

In case  $p = p_0$  we take  $w = 0$  so that the right-hand side in (3.1) becomes  $-hF(p_0, \Gamma_0) + K_0h^{1+\gamma}$ . Otherwise, by the choice  $w = -h^\delta(p_0 - p)/|p_0 - p|$ , the right-hand side in (3.1) becomes

$$\begin{aligned} & -h^{1/2+\delta}|p_0 - p| + K_4h^{1+\delta} - hF(p_0, \Gamma_0) + K_0h|p_0 - p| + K_0h^{1+\gamma} \\ & = |p_0 - p|(-h^{1/2+\delta} + K_0h) - hF(p_0, \Gamma_0) + K_4h^{1+\delta} + K_0h^{1+\gamma} \\ & \leq -hF(p_0, \Gamma_0) + (K_0 + K_4)h^{1+\min\{\gamma, \delta\}} \end{aligned}$$

for any sufficiently small  $h$  since there exists  $h'_3 \in (0, h_3]$  such that  $-h^{1/2+\delta} + K_0h \leq 0$  for all  $h \in (0, h'_3]$ .

Suppose that the minimum eigenvalue  $\mu$  of  $\Gamma_0 - \Gamma$  is less than  $-h^\gamma$ . Then take  $w \neq 0$  as an eigenvector with respect to  $\mu$  such that  $(p_0 - p)^\top w \leq 0$  and  $|w| = h^{-\kappa}$ . This choice yields

$$\begin{aligned} & \sqrt{h}(p_0 - p)^\top w + \frac{h}{2}w^\top (\Gamma_0 - \Gamma)w - hF(p, \Gamma) \\ & \leq \frac{h}{2}\mu|w|^2 - hF(p_0, \Gamma_0) + hK_0(|p - p_0| + |\Gamma - \Gamma_0|) \\ & \leq -\frac{h}{2}h^\gamma h^{-2\kappa} - hF(p_0, \Gamma_0) + 4K_0K_4h^{1-\delta} \leq -hF(p_0, \Gamma_0) + \frac{h^{-\delta}}{2}(-h^{1+\gamma-2\kappa+\delta} + 8K_0K_4h), \end{aligned}$$

and the right-hand side in the last inequality just above is at most  $-hF(p_0, \Gamma_0)$  for any sufficiently small  $h$  since there exists  $h''_3 \in (0, h'_3]$  such that  $-h^{1+\gamma-2\kappa+\delta} + 8K_0K_4h \leq 0$  for all  $h \in (0, h''_3]$ .

Therefore, we have proved that for any  $(p, \Gamma) \in \mathcal{D}_{h, \delta}$ ,

$$\inf_{w \in \mathcal{D}_h} \left[ \sqrt{h}(p_0 - p)^\top w + \frac{h}{2} w^\top (\Gamma_0 - \Gamma) w - hF(p, \Gamma) \right] \leq -hF(p_0, \Gamma_0) + Ch^{1+\beta}$$

for some  $\beta = \beta_\delta$ . Combining this with Taylor expansion of  $\phi$  up to the second term, we obtain

$$\sup_{(p, \Gamma) \in \mathcal{D}_{h, \delta}} \inf_{w \in \mathcal{D}_{h, \kappa}} \left[ \phi(x + \sqrt{h}w) - \sqrt{h}p^\top w - \frac{h}{2} w^\top \Gamma w - hF(p, \Gamma) \right] \leq -hF(p_0, \Gamma_0) + Ch^{1+\beta},$$

which completes the proof of the lemma.  $\square$

The function  $v^h$  is actually bounded with respect to  $h$  and  $x$ .

**Lemma 3.12.** *Under the assumptions imposed in Theorem 3.7, there exist  $h_4 \in (0, 1]$  and a positive constant  $K_5$  such that  $|v^h(t_k, x)| \leq K_5$  for  $k = 0, \dots, n$ ,  $h \leq h_4$ , and  $x \in \Omega$ .*

*Proof.* Assumption 3.6 and  $f \in C(\Omega)$  mean

$$|v^h(t_n, x)| \leq B_n, \quad x \in \Omega, \quad h \in (0, 1]$$

for some positive constant  $B_n$ . So suppose that for  $k \leq n-1$  there exists  $B_{k+1} > 0$  such that

$$|v^h(t_{k+1}, x)| \leq B_{k+1}, \quad x \in \Omega, \quad h \in (0, h_4]$$

with some  $h_4 \in (0, 1]$  to be determined below. To get a bound of  $v^h(t_k, \cdot)$ , rewrite  $v^h(t_k, x)$  as

$$v^h(t_k, x) = v^h(t_{k+1}, x) - hF(t_{k+1}, x; v^h(t_{k+1}, \cdot)) + hR_1^h(x) + hR_2^h(x) + hR_3^h(x),$$

where

$$\begin{aligned} R_1^h(x) &= (1 - \theta) \left( F(t_k, x; v^h(t_k, \cdot)) - I_{F(t_k, \cdot; v^h(t_k, \cdot)), X}(x) \right), \\ R_2^h(x) &= \theta \left( F(t_{k+1}, x; v^h(t_{k+1}, \cdot)) - I_{F(t_{k+1}, \cdot; v^h(t_{k+1}, \cdot)), X}(x) \right), \\ R_3^h(x) &= (1 - \theta) \left( F(t_{k+1}, x; v^h(t_{k+1}, \cdot)) - F(t_k, x; v^h(t_k, \cdot)) \right). \end{aligned}$$

Further, note that by Assumption 3.1,

$$\begin{aligned} (3.2) \quad & |F(t_{k+1}, x; v^h(t_{k+1}, \cdot)) - F(t_k, x; v^h(t_k, \cdot))| \\ & \leq |F_0(t_{k+1}, x; v^h(t_{k+1}, x)) - F_0(t_k, x; v^h(t_k, x))| \\ & \quad + K_0 |Dv^h(t_{k+1}, x) - Dv^h(t_k, x)| + K_0 |D^2v^h(t_{k+1}, x) - D^2v^h(t_k, x)|. \end{aligned}$$

Assumption 3.6, Lemma 3.10 and (3.2) then guarantee that  $\sum_{i=1}^3 \sup_{x \in \Omega} |R_i^h(x)|$  is bounded with respect to  $h$ . Thus Lemma 3.11 yields  $|v^h(t_k, x)| \leq |Q| + Ch$  where

$$Q = \sup_{(p, \Gamma) \in \mathcal{D}_{h, \delta}} \inf_{w \in \mathcal{D}_{h, \kappa}} \left[ v^h(t_{k+1}, x + \sqrt{h}w) - \sqrt{h}w^\top p - \frac{h}{2} w^\top \Gamma w - hF(t, x, v^h(t_{k+1}, x), p, \Gamma) \right].$$

Considering  $p = 0$  and  $\Gamma = 0$ , we see  $Q \geq -(1 + K_1 h)B_{k+1} - K_1 h$ .

To obtain an upper bound, observe

$$Q \leq B_{k+1} + \sup_{(p, \Gamma) \in \mathcal{D}_{h, \delta}} \inf_{w \in \mathcal{X}_{h, \kappa}} Q_{p, \Gamma, w},$$

where

$$Q_{p, \Gamma, w} = -\sqrt{h} w^\top p - \frac{h}{2} w^\top \Gamma w - hF(t, x, v^h(t_{k+1}, x), p, \Gamma).$$

Then we will show that for any  $(p, \Gamma) \in \mathcal{F}_{h, \delta}$  we can find  $w \in \mathcal{X}_{h, \kappa}$  satisfying  $Q_{p, \Gamma, w} \leq K_1 h B_{k+1} + Ch$ . So fix  $(p, \Gamma) \in \mathcal{F}_{h, \delta}$ . First assume that the minimum eigenvalue of  $-\Gamma$  is greater than or equal to  $-h^\gamma$ . If  $p = 0$  then we may take  $w = 0$ , leading to  $Q_{p, \Gamma, w} \leq -hF(t, x, v^h(t_{k+1}, x), 0, h^\gamma I) \leq K_1 h + K_1 h B_{k+1} + K_1 h^{1+\gamma}$ . Otherwise, take  $w = h^\delta p / |p|$ . Then we see

$$\begin{aligned} Q_{p, \Gamma, w} &\leq -h^{(1/2)+\delta} |p| + \frac{h^{1+\delta}}{2} + K_1 h(1 + B_{k+1}) + K_1 h |p| + K_1 h^{1+\gamma} \\ &\leq |p|(-h^{(1/2)+\delta} + K_1 h) + Ch + K_1 h B_{k+1} \leq Ch + K_1 h B_{k+1} \end{aligned}$$

since  $-h^{(1/2)+\delta} + K_1 h \leq 0$  for  $h \in (0, h'_4]$  with some  $h'_4 \in (0, h_3]$ .

Next assume that the minimum eigenvalue of  $-\Gamma$  is less than  $-h^\gamma$ . Then take  $w$  to be the corresponding eigenvector satisfying  $-p^\top w \leq 0$  and  $|w| = h^{-\kappa}$ . This choice leads to

$$Q_{p, \Gamma, w} \leq -\frac{h^{1+\gamma-2\kappa}}{2} + K_1 h(1 + B_{k+1}) + 2K_1 h^{1-\delta} \leq K_1 h(1 + B_{k+1})$$

since there exists  $h_4 \in (0, h'_4]$  such that  $-h^{1+\gamma-2\kappa+\delta} + 4K_1 h \leq 0$  for  $h \in (0, h_4]$ .

Therefore we deduce that  $|Q| \leq (1 + K_1 h)B_{k+1} + Ch$  for  $h \leq h_4$ . Denoting the right-hand side by  $B_k$ , we obtain the sequence  $\{B_k\}$  satisfying  $B_k = (1 + K_1 h)B_{k+1} + Ch$ . By a routine argument we have  $B_k \leq e^{TK_1} B_n + C e^{TK_1}$  for all  $k$ . Thus the lemma follows.  $\square$

*Proof of Theorem 3.7.* We adopt the viscosity solution method as stated in [1]. To this end, we set  $v^h(t, x) = v(t, x)$  for  $(t, x) \in [0, T] \times (\mathbb{R}^d \setminus \Omega)$  and consider

$$\bar{v}(t, x) = \limsup_{\substack{s \rightarrow t, y \rightarrow x \\ h \searrow 0}} v^h(s, y), \quad (t, x) \in [0, T] \times \mathbb{R}^d$$

to show that  $\bar{v}$  is a viscosity subsolution of (1.1). Lemma 3.12 implies that  $\bar{v}$  is finite on  $[0, T] \times \mathbb{R}^d$ .

Fix  $(t, x) \in [0, T] \times \Omega$  and let  $\varphi \in C^3([0, T] \times \Omega)$  such that  $\bar{v} - \varphi$  has a local maximum at  $(t, x)$ . Then, take  $r > 0$  such that

$$(\bar{v} - \varphi)(s, y) \leq (\bar{v} - \varphi)(t, x), \quad (s, y) \in B_r(t, x),$$

where  $B_r(t, x)$  is the closed ball centered at  $(t, x)$  with radius  $r$ , and that  $B_r(t, x) \subset [0, T] \times \Omega$ . Next, for  $(s, y) \in B_r(t, x)$  set

$$\tilde{\varphi}(s, y) = \varphi(s, y) - (\varphi(t, x) - \bar{v}(t, x)) + |s - t|^2 + |y - x|^2.$$

It follows that  $\bar{v}(t, x) = \bar{\varphi}(t, x)$  and that  $(t, x)$  is a strict maximum of  $\bar{v} - \bar{\varphi}$  on  $B_r(t, x)$ . By abuse of notation, we write  $\varphi$  for  $\bar{\varphi}$ .

By definition of  $\bar{v}$ , there exist  $h_m$  and  $(\tilde{s}_m, \tilde{y}_m) \in B_r(t, x)$  such that, as  $m \rightarrow \infty$ ,

$$h_m \rightarrow 0, \quad (\tilde{s}_m, \tilde{y}_m) \rightarrow (t, x), \quad v^{h_m}(\tilde{s}_m, \tilde{y}_m) \rightarrow \bar{v}(t, x).$$

Take  $s_m$  and  $y_m$  so that  $s_m = ih_m$  for some  $i = i_m = 0, \dots, n-1$  and that

$$(3.3) \quad (v^{h_m} - \varphi)(s_m, y_m) \geq \sup_{(s, y) \in B_r(t, x)} (v^{h_m} - \varphi)(s, y) - h_m^{3/2}.$$

Moreover, the sequence  $(s_m, y_m)$ ,  $m \geq 1$ , can be taken from the bounded set  $B_r(t, x)$ , so there exists a limit point  $(\tilde{t}, \tilde{x}) \in B_r(t, x)$  possibly along a subsequence. Thus, denoting  $c_m = (v^{h_m} - \varphi)(s_m, y_m)$ , we have

$$0 = (\bar{v} - \varphi)(t, x) = \lim_{m \rightarrow \infty} (v^{h_m} - \varphi)(\tilde{s}_m, \tilde{y}_m) \leq \liminf_{m \rightarrow \infty} c_m \leq \limsup_{m \rightarrow \infty} c_m \leq (\bar{v} - \varphi)(\tilde{t}, \tilde{x}).$$

Since  $(t, x)$  is a strict maximum, we deduce that  $(\tilde{t}, \tilde{x}) = (t, x)$ . Therefore, it follows that  $(s_m, y_m) \rightarrow (t, x)$  and  $c_m \rightarrow 0$ . By (3.3), for any  $y$  near  $x$ ,

$$(3.4) \quad \varphi(s_m + h_m, y) + c_m + h_m^{3/2} \geq v^{h_m}(s_m + h_m, y).$$

Now, by Lemma 3.10, we have

$$\sum_{i=0}^2 |D^i v^{h_m}(s_m + h_m, y_m) - D^i v^{h_m}(s_m, y_m)| \rightarrow 0,$$

as  $m \rightarrow \infty$ . In particular,

$$\lim_{m \rightarrow \infty} v^{h_m}(s_m + h_m, y_m) = \lim_{m \rightarrow \infty} v^{h_m}(s_m, y_m) = \varphi(t, x).$$

Also, by Assumption 3.1,

$$(3.5) \quad \begin{aligned} & |F(s_m + h_m, y_m; v^{h_m}(s_m + h_m, \cdot)) - F(s_m, y_m; v^{h_m}(s_m, \cdot))| \\ & \leq |F_0(s_m + h_m, y_m, v^{h_m}(s_m + h_m, y_m)) - F_0(t, x, \varphi(t, x))| \\ & \quad + |F_0(t, x, \varphi(t, x)) - F_0(s_m, y_m, v^{h_m}(s_m, y_m))| \\ & \quad + K_0 |Dv^{h_m}(s_m + h_m, y_m) - Dv^{h_m}(s_m, y_m)| + |D^2 v^{h_m}(s_m + h_m, y_m) - D^2 v^{h_m}(s_m, y_m)|. \end{aligned}$$

As in the proof of Lemma 3.12, rewrite  $v^h(s_m, y_m)$  as

$$(3.6) \quad \begin{aligned} v^{h_m}(s_m, y_m) &= v^{h_m}(s_m + h_m, y_m) - h_m F(s_m + h_m, y_m; v^{h_m}(s_m + h_m, \cdot)) \\ &\quad + h_m R_1^m + h_m R_2^m + h_m R_3^m, \end{aligned}$$

where

$$\begin{aligned} R_1^m &= (1 - \theta) \left( F(s_m, y_m; v^{h_m}(s_m, \cdot)) - I_{F(s_m, \cdot; v^{h_m}(s_m, \cdot)), X}(y_m) \right), \\ R_2^m &= \theta \left( F(s_m + h_m, y_m; v^{h_m}(s_m + h_m, \cdot)) - I_{F(s_m + h_m, \cdot; v^{h_m}(s_m + h_m, \cdot)), X}(y_m) \right), \\ R_3^m &= (1 - \theta) \left( F(s_m + h_m, y_m; v^{h_m}(s_m + h_m, \cdot)) - F(s_m, y_m; v^{h_m}(s_m, \cdot)) \right). \end{aligned}$$



Assumption 3.6, Lemma 3.10 and (3.5) guarantee  $R_1^m, R_2^m, R_3^m \rightarrow 0$  as  $m \rightarrow \infty$ . With the representation (3.6), we apply Lemma 3.11 for the family  $\{v^h(s_m + h, \cdot), \varphi(s_m + h, \cdot)\}_{h \in (0,1], m \geq 1}$  and use the inequality (3.4) to get, for any sufficiently large  $m$ ,

$$\begin{aligned}
& v^{h_m}(s_m, y_m) \\
& \leq \sup_{(p, \Gamma) \in \mathcal{D}_{h_m, \delta}} \inf_{w \in \mathcal{X}_{h_m, \kappa}} \left[ v^{h_m}(s_m + h_m, y_m + \sqrt{h_m} w) - \sqrt{h_m} p^\top w - \frac{h_m}{2} w^\top \Gamma w \right. \\
& \quad \left. - h_m F(s_m + h_m, y_m, v^{h_m}(s_m + h_m, y_m), p, \Gamma) \right] + h_m R_1^m + h_m R_2^m + h_m R_3^m + C h_m^{1+\beta} \\
& \leq \sup_{p, \Gamma} \inf_w \left[ \varphi(s_m + h_m, y_m + \sqrt{h_m} w) - \sqrt{h_m} p^\top w - \frac{h_m}{2} w^\top \Gamma w \right. \\
& \quad \left. - h_m F(s_m + h_m, y_m, v^{h_m}(s_m + h_m, y_m), p, \Gamma) \right] + c_m + h_m^{3/2} + h_m R_1^m + h_m R_2^m + h_m R_3^m + C h_m^{1+\beta} \\
& \leq \varphi(s_m + h_m, y_m) - h_m F(s_m + h_m, y_m, v^{h_m}(s_m + h_m, y_m), D\varphi(s_m + h_m, y_m), D^2\varphi(s_m + h_m, y_m)) \\
& \quad + c_m + h_m^{3/2} + h_m R_1^m + h_m R_2^m + h_m R_3^m + C h_m^{1+\beta}.
\end{aligned}$$

This and  $v^{h_m}(s_m, y_m) = c_m + \varphi(s_m, y_m)$  imply

$$\begin{aligned}
(3.7) \quad & -\frac{1}{h_m} (\varphi(s_m + h_m, y_m) - \varphi(s_m, y_m)) \\
& + F(s_m + h_m, y_m, v^{h_m}(s_m + h_m, y_m), D\varphi(s_m + h_m, y_m), D^2\varphi(s_m + h_m, y_m)) \leq o(1)
\end{aligned}$$

for any sufficiently large  $m$ . Letting  $m \rightarrow \infty$ , we arrive at

$$(3.8) \quad -\partial_t \varphi(t, x) + F(t, x, \bar{v}(t, x), D\varphi(t, x), D^2\varphi(t, x)) \leq 0.$$

Thus the subsolution property at  $(t, x)$  follows.

In the case  $(t, x) \in \{T\} \times \mathbb{R}^d$ , from the definition of  $v^h$  and Assumption 3.6 we have  $\bar{v}(t, x) = f(x)$ . Thus the subsolution property immediately follows.

Next consider the case  $(t, x) \in [0, T) \times \partial\Omega$ . As in the first part of the proof, we can take the sequence  $(h_m, s_m, y_m)$ ,  $m \geq 1$ , satisfying (3.4) and  $(s_m, y_m) \rightarrow (t, x)$ . Moreover,

$$v^{h_m}(s_m, y_m) = c_m + \varphi(s_m, y_m) \rightarrow \varphi(t, x) = \bar{v}(t, x).$$

Then, if there exists  $m_0 \geq 1$  such that  $y_m \in \mathbb{R}^d \setminus \Omega$  for all  $m \geq m_0$ , we see

$$v^{h_m}(s_m, y_m) = v(s_m, y_m) \rightarrow v(t, x), \quad m \rightarrow \infty.$$

Thus the subsolution property follows. Otherwise, there exists a subsequence  $\{y_{m_j}\}$  such that  $y_{m_j} \in \Omega$  and  $y_{m_j} \rightarrow x$ ,  $j \rightarrow \infty$ . With this sequence we obtain the inequality (3.7) with  $(h_m, s_m, y_m)$  replaced by  $(h_{m_j}, s_{m_j}, y_{m_j})$ . Then letting  $j \rightarrow \infty$ , we obtain (3.8) at  $(t, x) \in [0, T) \times \partial\Omega$ .

By similar arguments, we can show that

$$\underline{v}(t, x) = \liminf_{\substack{s \rightarrow t, y \rightarrow x \\ h \searrow 0}} v^h(s, y), \quad (t, x) \in [0, T] \times \mathbb{R}^d$$

is a viscosity supersolution of (1.1). The comparison principle now implies that  $\bar{v} \leq \underline{v}$ . However, by definition,  $\bar{v} \geq \underline{v}$ . Hence we obtain  $\bar{v} = \underline{v}$ , as asserted.  $\square$

## 4 A numerical example

Here we consider the following two-dimensional deterministic KPZ equation

$$\begin{cases} \partial_t v + \frac{1}{2} \text{tr}(D^2 v) + \frac{1}{2} |Dv|^2 = 0, \\ v(1, x) = f(x). \end{cases}$$

By Cole-Hopf transformation (see, e.g., Evans [6]), the unique solution is represented as

$$v(t, x) = \log \mathbb{E}[\exp(f(x + W_{1-t}))], \quad (t, x) \in [0, 1] \times \mathbb{R}^2,$$

where  $\{W_t\}_{0 \leq t \leq 1}$  is a 2-dimensional standard Brownian motion and  $\mathbb{E}$  is the expectation operator on a probability space.

We examine the case of the terminal data given by

$$f(x_1, x_2) = \cos(x_1) \cos(x_2)$$

and compute the solution in  $\{0\} \times [-\pi/4, \pi/4]^2$  by our collocation method with  $\theta = 1$  and Gaussian RBF. We use the equi-spaced grids on  $[-\pi/2, \pi/2]^2$  consisting of  $N$  points for the set  $X$  of the data sites. Notice that we take the larger region  $[-\pi/2, \pi/2]^2$  to expect a better performance near the boundary of  $[-\pi/4, \pi/4]^2$ . The adjustable parameter  $\alpha$  for the kernel is set as  $\alpha = 1/\varepsilon^2$  where  $\varepsilon$  is the Euclidean norm between the  $N$  points in  $[-\pi/2, \pi/2]^2$ . As the benchmark, the exact solution  $v(0, x)$  is estimated by the Monte-Carlo method with  $10^6$  samples. Table 4 shows the root mean square errors and the maximum errors, defined by

$$\text{RMS error} = \sqrt{\frac{1}{625} \sum_{x \in X_0} |v^{h,N}(0, x) - v(0, x)|^2}, \quad \text{Max error} = \max_{x \in X_0} |v^{h,N}(0, x) - v(0, x)|,$$

respectively, where  $X_0$  is the set of evaluation points consisting of  $25^2$  equi-spaced grid points in  $[-\pi/4, \pi/4]^2$ .

	$N = 9$	$N = 16$	$N = 25$
RMS error	0.0409921405159	0.0128968947645	0.00273912837108
Max error	0.0469188454774	0.0166425004758	0.00490893471041

Table 4.1: RMS and Max errors for several choices of  $N$  with  $h = 10^{-2}$

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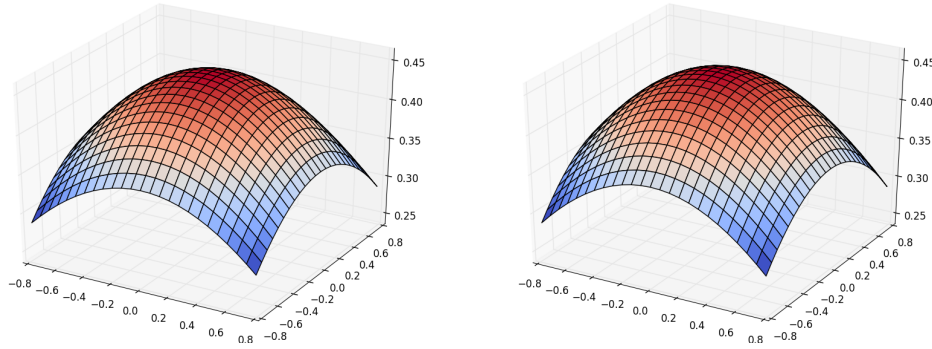


Figure 4.1: The analytical solution (left) and the numerical solution (right) with  $h = 10^{-2}$  and  $N = 25$ .

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